

# Line Sources in Brans-Dicke Theory of Gravity

F. Dahia\* and C. Romero†

*Departamento de Física*

*Universidade Federal da Paraíba*

*Cx. Postal 5008, 58059-970, João Pessoa, PB, Brazil*

We investigate how the gravitational field generated by line sources can

be characterized in Brans-Dicke theory of gravity. Adapting an approach previously developed by Israel who solved the same problem in general relativity we show that in Brans-Dicke theory's case it is possible to work out the field equations which relate the energy-momentum tensor of the source to the scalar field, the coupling constant  $\omega$  and the extrinsic curvature of a tube of constant geodesic radius centered on the line in the limit when the radius shrinks to zero. In this new scenario two examples are considered and an account of the Gundlach and Ortiz solution is included. Finally, a brief discussion of how to treat thin shells in Brans-Dicke theory is given.

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\*e-mail: fdahia@fisica.ufpb.br

†e-mail: cromero@fisica.ufpb.br

## I. INTRODUCTION

Topological defects such as cosmic strings, domain walls, monopoles and textures [1] have become recently a very significant part of current theoretical physics research. Predicted by GUT models these structures are now considered to be viable candidates for explaining a number of observational phenomena in Astrophysics and Cosmology. Cosmic strings, in particular, which are thought to have been generated in the early Universe, during a phase transition, might be responsible for gravitational lensing [2] and galaxy formation [3], among other physical phenomena. Naturally, from a theoretical standpoint, the entire subject was first investigated in the context of general relativity. Recently, however, due to a renewed interest in Brans-Dicke theory, mainly in connection with the inflation problem, a number of authors [4–11] have started considering cosmic strings, domain walls and monopoles in scalar-tensor theories as well as in dilaton gravity. As a result of these investigations it has been shown that scalar-tensor theories lead to the prediction of new and different phenomena concerning the gravitational effects produced by topological defects. As an example, let us mention the appearance of gravitational forces exerted by a global monopole on the matter around it, an effect predicted by Brans-Dicke theory and which is absent in the case of general relativity's monopole [1,8].

Historically one could say that cosmic strings, the most studied of all topological defects, entered the cosmological scenario through the work by Vilenkin [12] who solved the Einstein field equations for a matter distribution corresponding to an infinite, static, straight string. Vilenkin's solution, which was obtained using the weak field approximation of general relativity depicts the gravitational field generated by the string as described by a conical space-time whose angular deficit is related to the mass density. Exact solutions of a simple model of the string were found later by Gott [2] and Hiscock [13] independently and also correspond to conical space-time.

The matter distribution which represents a cosmic string is a very particular case of an idealization of realist matter distribution which may be generally called line sources. Long

before cosmic strings had become known in the literature the problem of how to characterize line sources in general relativity and how to compute the gravitational field generated by such structures was already discussed by Israel [14] with a high degree of generality. As shown by Israel, this is a complex problem and as yet there is no simple prescription of how to characterize physically an arbitrary line source. However, for a wide class of line sources one can work out through the field equations a relation between the line energy-momentum tensor and the extrinsic curvature of a tube of matter enclosing the source in the limit when the radius of the tube tends to zero.

The purpose of this paper is to consider the method employed by Israel concerning line sources in general relativity and extend it to investigate the same problem in Brans-Dicke theory of gravity. Thus, in section II we give a general description of the so-called simple line sources in Brans-Dicke theory. In section III we attempt to assign a “line energy-momentum tensor” density to a simple line. Section IV deals with applications and includes a discussion of cosmic strings in Brans-Dicke theory. Finally, a brief account of thin shells or surface layers is given in section V.

## II. GENERAL CHARACTERIZATION OF LINE SOURCES IN BRANS-DICKE THEORY OF GRAVITY

Given that lines are essentially idealizations of realistic matter distributions, we start by partially characterizing a line source as a “singularity” that can be enclosed in a tube of arbitrary small radius. Then, we proceed to give the following definitions:

A two-dimensional submanifold  $L$  embedded in space-time is called a “line” if the following conditions hold:

- (i) There exists a neighborhood  $N$  of  $L$  such that each point  $p \in N$  can be connected to  $L$  by a spacelike curve of finite length and for each  $p \in N$  there exists a minimal length curve, which is a geodesic of length  $\rho(p)$ , connecting  $p$  to  $L$ . This function  $\rho(p)$  defines the geodesic radius of  $p$  and may be taken as one of the coordinates of the space-times points in

the neighborhood  $N$ . It can be shown that these geodesics are orthogonal to the hypersurface  $\Sigma$  defined by the equation  $\rho = \text{const}$ .

(ii) The hypersurfaces  $\Sigma$  are tridimensional timelike submanifolds embedded in space-time (“three-cylinders”) with topology  $S^1 \times S^1 \times \mathbb{R}$  or  $S^1 \times \mathbb{R}^2$  depending on the lines  $L$  being closed or not.

(iii) Each three-cylinder  $\Sigma$  may be covered by a congruence of simple nonreducible closed spacelike curves whose length tends to zero as they are Lie-transported inward to  $L$  along radial geodesics.

Clearly, the definition above naturally suggests the choice of the following coordinate system to cover the region  $N$ . We choose  $\rho$  as a radial coordinate and, taking into account property (iii), choose an angular coordinate  $\varphi$  ( $0 < \varphi < 2\pi$ ) that acts as a parametrization of the congruence of curves which covers the family of hypersurfaces  $\Sigma$ . Then we introduce two more coordinates,  $z$  and  $t$ , in one of the three-cylinders  $\Sigma$  and Lie-transport  $(z, t)$  along radial geodesics to cover the entire region  $N$ . In this way, we end up with a Gaussian coordinate system in terms of which the metric takes the form

$$ds^2 = d\rho^2 + g_{ij}(\rho, x^k) dx^i dx^j, \quad (1)$$

where the Latin indices coordinates  $x^i$  denotes  $(t, \varphi, z)$ , with  $t \in \mathbb{R}$ ,  $0 < \varphi < 2\pi$  and the range of  $z$  depends on the three-cylinders topology.

Now, from the form (1) of the metric written in the Gaussian coordinates  $(\rho, t, \varphi, z)$  we can easily calculate the extrinsic curvature  $K_{ij}$  of the three-cylinders  $\Sigma$  along with the corresponding density  $\mathcal{K}_{ij}$  to obtain

$$K_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial \rho}, \quad \mathcal{K}_j^i = \sqrt{-g} K_j^i. \quad (2)$$

On the other hand, the foliation  $\rho = \text{const}$  allows us to decompose the Ricci tensor in terms of the extrinsic and intrinsic curvature of the three-cylinders  $\Sigma$ .

At this point let us consider Brans-Dicke field equations in the form

$$R_\nu^\mu + W_\nu^\mu = -\frac{8\pi}{\phi} \left( T_\nu^\mu - \frac{1}{2} \delta_\nu^\mu f(\omega) T \right) \quad (3a)$$

$$\square\phi = \frac{8\pi}{3+2\omega} T, \quad (3b)$$

where  $\phi$  is the scalar field,  $\omega$  is the coupling constant,  $T_\nu^\mu$  is the energy-momentum tensor of the matter fields,  $T = T_\mu^\mu$ , and we have defined

$$W_\nu^\mu \equiv \frac{\omega}{\phi^2} \phi^{,\mu} \phi_{,\nu} + \frac{1}{\phi} \phi^{;\mu}_{;\nu} \quad (4a)$$

$$f(\omega) \equiv \frac{2+2\omega}{3+2\omega}. \quad (4b)$$

One can show that the field equations (3a) and (3b) may be decomposed into

$$\frac{\partial}{\partial\rho} (\phi K_j^i) + \phi \sqrt{-g} (({}^{(3)}R_j^i + {}^{(3)}W_j^i)) = -8\pi \sqrt{-g} \left( T_j^i - \frac{1}{2} \delta_j^i f(\omega) T \right) \quad (5a)$$

$$K_{,j} - {}^{(3)}\nabla_i K_j^i - \frac{\phi_{,i}}{\phi} K_j^i + \frac{\phi_{,\rho},j}{\phi} + \omega \frac{\phi_{,\rho}\phi_{,j}}{\phi^2} = -\frac{8\pi}{\phi} T_j^\rho \quad (5b)$$

$$K_{ij} K^{ij} - K^2 + \omega \left( \frac{\phi_{,\rho}}{\phi} \right)^2 - 2K \left( \frac{\phi_{,\rho}}{\phi} \right) - {}^{(3)}R - {}^{(3)}W = -\frac{16\pi}{\phi} T_\rho^\rho \quad (5c)$$

$$\frac{\partial}{\partial\rho} (\sqrt{-g} \phi_{,\rho}) + \sqrt{-g} ({}^{(3)}\nabla^2 \phi) = \frac{8\pi}{3+2\omega} \sqrt{-g} T, \quad (5d)$$

where  $K = K_i^i$ ,  ${}^{(3)}W = W_i^i$ ,  ${}^{(3)}R_j^i$  and the operator  ${}^{(3)}\nabla_i$  denote the Ricci tensor and the covariant derivative associated with the three-metric  $g_{ij}$ , respectively.

At this stage let us define the concept of simple line in Brans-Dicke theory:

A line  $L$  is called a *simple line* if the following additional conditions hold:

(iv) The intrinsic curvature density of the cylinders  $\rho = const$  converges less rapidly than the term  $\frac{1}{\phi\rho}$ , i.e.,  $\lim_{\rho\rightarrow 0} (\rho\phi \sqrt{-g} {}^{(3)}R_j^i) = 0$ . This condition, called normal-dominated convergence, is required to be fulfilled also by  ${}^{(3)}W_j^i$ , i.e.,  $\lim_{\rho\rightarrow 0} (\rho\phi \sqrt{-g} {}^{(3)}W_j^i) = 0$ . Also, for the term  ${}^{(3)}\nabla^2 \phi$  we should have  $\lim_{\rho\rightarrow 0} (\rho \sqrt{-g} {}^{(3)}\nabla^2 \phi) = 0$ . (As we shall see, these conditions imply the existence of the following limits:  $\lim_{\rho\rightarrow 0} (\phi K_j^i) = \mathcal{C}_j^i(t, \varphi, z)$  and  $\lim_{\rho\rightarrow 0} (\sqrt{-g} \phi_{,\rho}) = \ell(t, \varphi, z)$ .)

(v) We assume that the closed curves of condition (iii) can be chosen in such a way that

$$\frac{\partial \mathcal{C}_j^i}{\partial \varphi} = 0 \quad (6)$$

(vi)  $C_j^i$  can be put in the diagonal form  $C_j^i(t, z) = diag(\alpha, \beta, \gamma)$  by a suitable choice of coordinates  $z$  and  $t$ , where  $\alpha, \beta$  and  $\gamma$  are functions of  $t$  and  $z$ .

Due to the presence of the line  $L$  the energy-momentum tensor  $T_\nu^\mu$ , which describes the total content of matter distributed over the whole space-time, may be decomposed into two parts:  $T_\nu^\mu = (T_{reg})_\nu^\mu + \mathcal{L}_\nu^\mu \delta(\rho)$ , where  $(T_{reg})_\nu^\mu$  accounts for the regular matter surrounding the line and  $\mathcal{L}_\nu^\mu$  describes the physical properties of the matter concentrated on  $L$ .

Now, if we multiply Eq. (5a) by  $\rho$ , take the limit  $\rho \rightarrow 0$  and assume that  $(T_{reg})_\nu^\mu$  satisfies the normal dominated convergence condition ( $\lim_{\rho \rightarrow 0} [\sqrt{-g} (T_{reg})_\nu^\mu] = 0$ ), we obtain

$$\lim_{\rho \rightarrow 0} \rho \frac{\partial}{\partial \rho} (\phi \mathcal{K}_j^i) = 0. \quad (7)$$

Therefore for small values of  $\rho$  we have asymptotically  $\phi \mathcal{K}_j^i \simeq \mathcal{C}_j^i + \mathcal{O}_j^i$ , where the order of  $\mathcal{O}_j^i$  is greater than that of  $\rho^\delta$ , with  $\delta > 0$ . It follows that

$$\lim_{\rho \rightarrow 0} \phi \mathcal{K}_j^i = \mathcal{C}_j^i, \quad (8)$$

where it must be remembered that  $\mathcal{C}_j^i$  does not depend on  $\varphi$  by virtue of condition (v).

Likewise, Eq. (5d) yields

$$\lim_{\rho \rightarrow 0} (\sqrt{-g} \phi_{,\rho}) = \ell(t, z), \quad (9)$$

where for consistency we are assuming that  $\frac{\partial \ell}{\partial \varphi} = 0$ .

It is worth mentioning that the existence of these limits determines the asymptotic equations (when  $\rho \rightarrow 0$ ) for the metric and the scalar field. Indeed, from the definition of extrinsic curvature we have in Gaussian coordinates  $\mathcal{K}_j^i = \frac{1}{2} \sqrt{-g} g^{ik} g_{kj,\rho}$ ; whence from (8) for  $\rho \rightarrow 0$  it follows that

$$\frac{1}{2} \phi \sqrt{-g} g^{ik} g_{kj,\rho} = \mathcal{C}_j^i. \quad (10)$$

Taking the trace of this equation we get

$$\phi \frac{\partial \sqrt{-g}}{\partial \rho} = \mathcal{C}. \quad (11)$$

Clearly, the asymptotic behavior of the term  $\phi \sqrt{-g}$  is obtained by taking together (9) and (11). Thus, assuming that  $\lim_{\rho \rightarrow 0} \phi \sqrt{-g} = 0$  we are led to the equation

$$\phi\sqrt{-g} = (\mathcal{C} + \ell)\rho, \quad (12)$$

for  $\rho \rightarrow 0$ . Now, substituting (12) into (10) and considering that  $\mathcal{C}_j^i$  has no degenerate eigenvalue we obtain, for  $\rho \rightarrow 0$ ,

$$g_{ij} = \text{diag}(h_{tt}(t, z)\rho^{2a}, h_{\varphi\varphi}(t, z)\rho^{2b}, h_{zz}(t, z)\rho^{2c}), \quad (13)$$

where  $h_{tt}$ ,  $h_{\varphi\varphi}$ , and  $h_{zz}$  are arbitrary functions of  $t$  and  $z$ , and  $a(t, z)$ ,  $b(t, z)$  and  $c(t, z)$  are defined by the equations  $a = \frac{\mathcal{C}_t^t}{\mathcal{C} + \ell}$ ,  $b = \frac{\mathcal{C}_\varphi^\varphi}{\mathcal{C} + \ell}$  and  $c = \frac{\mathcal{C}_z^z}{\mathcal{C} + \ell}$ . Let us note that in the degenerate case  $\mathcal{C}_t^t = \mathcal{C}_z^z$  by transforming the coordinates  $t$  and  $z$  it is still possible to put the metric in the form (13).

As to the scalar field, from (13) we have  $\sqrt{-g} = \sqrt{-h}\rho^{\frac{c}{\mathcal{C} + \ell}}$ , with  $h = h_{tt}h_{\varphi\varphi}h_{zz}$ . Then, from (12) we find directly that for  $\rho \rightarrow 0$  we have

$$\phi(\rho, t, z) = \psi(z, t)\rho^d, \quad (14)$$

with  $d \equiv \frac{\ell}{\mathcal{C} + \ell}$  and the function  $\psi(z, t)$  is defined by  $\psi(z, t)\sqrt{-h} = \mathcal{C} + \ell$ .

Now, if we multiply the field equation (5c) by  $(\phi\sqrt{-g})^2$  and then take the limit  $\rho \rightarrow 0$  we readily obtain

$$\mathcal{C}_j^i\mathcal{C}_i^j - \mathcal{C}^2 + \omega\ell^2 - 2\mathcal{C}\ell = 0. \quad (15)$$

In terms of the functions  $a$ ,  $b$ ,  $c$  and  $d$  we have following constraints:

$$a + b + c + d = 1 \quad (16)$$

and

$$a^2 + b^2 + c^2 + (1 + \omega)d^2 = 1. \quad (17)$$

Let us note that if  $d \rightarrow 0$  more quickly than  $\omega^{-\frac{1}{2}}$  when  $\omega \rightarrow \infty$ , then in this limit the General Relativistic case is recovered [14].

These results give a description of the space-time geometry in the exterior region near the simple line. However, to characterize completely the energy-momentum tensor of the line we need to consider the “interior” of the line, i.e., the matter tubes whose idealization is pictured by the line.

### III. MODELS OF MATTER IN THE INTERIOR OF THE LINE

It seems reasonable to define the energy-momentum tensor of a simple line source by the expression

$$\mathcal{L}_\nu^\mu = \lim_{\varepsilon \rightarrow 0} \int_0^\varepsilon \int_0^{2\pi} T_\nu^\mu \sqrt{-g} d\rho d\varphi. \quad (18)$$

It is worth mentioning, however, that this definition may not be useful to any kind of matter distribution concentrated along the line. In other words, the matter distribution should satisfy some conditions. For example, the exterior space-time generated by the matter distribution must be compatible with the geometry of simple lines discussed in the previous section, whereas in the interior of the tube matter must possess some special physical properties.

The first condition to be required refers to “axial symmetry”. Thus, in the interior of the tube we impose  $\frac{\partial T_\nu^\mu}{\partial \varphi} = 0$ ; whence it must follow that  $\frac{\partial \phi}{\partial \varphi} = 0$ . A second condition on the energy-momentum tensor requires that the radial pressure component  $T_\rho^\rho$  be much less than the other components of  $T_\nu^\mu$ . Further, let us admit that there exists a coordinate system in which the interior metric has the form

$$ds^2 = d\rho'^2 + g_{mn} dx^m dx^n + g_{\varphi\varphi} d\varphi^2, \quad (19)$$

where  $x^m = (t, z)$ ,  $0 \leq \rho' \leq \varepsilon$  and  $\frac{\partial g_{ij}}{\partial \varphi} = 0$  (due to the fact that  $\frac{\partial T_\nu^\mu}{\partial \varphi} = 0$ ). It is important to note that we are assuming that the internal coordinates  $(t, z, \varphi)$  may be taken continuous at  $\rho' = \varepsilon$  by Lie-transporting the exterior coordinates inward along radial geodesics [14]. On the other hand,  $\rho$  (the external radial coordinate) and  $\rho'$  may not be continuous on  $\rho' = \varepsilon$ , i.e., at the boundary of tube these coordinates may have different values. While  $\rho' = \varepsilon$ , a relation  $\rho = \rho(\varepsilon)$  may be obtained from the continuity of the metric.

If we assume that the metric is regular near the tube axis  $L$ , then we must have  $g_{\varphi\varphi} = \rho'^2$  when  $\rho' \rightarrow 0$ . Moreover, the bidimensional metric  $g_{mn}$  must be invertible, that is,  $\sqrt{-{}^{(2)}g} \neq 0$  at  $\rho' = 0$ . As to Brans-Dicke scalar field it is natural to require that  $\phi$  is smooth on the

axis, hence we assume that  $\lim_{\rho' \rightarrow 0} (\phi_{int}) = \eta(t, z) \neq 0$ . Further assumptions are related to convergence and are given by  $\lim_{\varepsilon \rightarrow 0} \left( \varepsilon \phi \sqrt{-g}^{(3)} R_j^i \Big|_\xi \right) = 0$ ,  $\lim_{\varepsilon \rightarrow 0} \left( \varepsilon \phi \sqrt{-g}^{(3)} W_j^i \Big|_\xi \right) = 0$  and  $\lim_{\varepsilon \rightarrow 0} \left( \varepsilon \sqrt{-g}^{(3)} \nabla^2 \phi_{int} \Big|_\xi \right) = 0$ , for every  $\xi$  inside the interval  $0 \leq \xi \leq \varepsilon$ . Then, taking into account these conditions and integrating the field equations (5a) and (5d) with respect to  $\rho$ , we obtain

$$\lim_{\varepsilon \rightarrow 0} \phi \mathcal{K}_j^i \Big|_0^\varepsilon = -4 \left( \mathfrak{L}_j^i - \frac{1}{2} \delta_j^i f(\omega) \mathfrak{L} \right) \quad (20a)$$

$$\lim_{\varepsilon \rightarrow 0} \sqrt{-g} \phi_{,\rho'} \Big|_0^\varepsilon = \frac{4}{3 + 2\omega} \mathfrak{L} \quad (20b)$$

The terms evaluated at  $\rho' = \varepsilon$  can be computed by using the exterior solution since we are assuming continuous junction between the external and internal solutions. However, for  $\rho' = 0$  these terms depend upon the internal solution and cannot be determined exactly, hence must be estimated somehow. In fact, such estimates are possible by virtue of some conditions we have assumed previously.

The components of the extrinsic curvature  $K_j^i$  which depend on  $t$  and  $z$  only are finite quantities due to the fact that the axis  $L$  is, by assumption, a smooth surface. Then, since  $\phi \sqrt{-g} \rightarrow \eta \sqrt{-{}^{(2)}g} \rho'$ , for  $\rho' \rightarrow 0$ , we have

$$\phi \mathcal{K}_n^m \Big|_0 = 0, \quad (21)$$

with  $m, n = t, z$ . To calculate the term  $\phi \mathcal{K}_\varphi^\varphi \Big|_0$  we just note that in the interior of the tube we have

$$\phi \mathcal{K}_\varphi^\varphi \Big|_0 = \eta \sqrt{-{}^{(2)}g} \Big|_0. \quad (22)$$

On the other hand, from (5c) we can derive the following equation:

$$K_\varphi^\varphi = -\frac{1}{2} \frac{\left[ 2 \det(K_n^m) + K_m^m \frac{\phi_{,\rho'}}{\phi} - \frac{1}{2} \omega \left( \frac{\phi_{,\rho'}}{\phi} \right)^2 + {}^{(3)}R + {}^{(3)}W + 16\pi T_{\rho'}^{\rho'} \right]}{K_m^m + \left( \frac{\phi_{,\rho'}}{\phi} \right)} \quad (23)$$

Since the cylinders  $\rho' = \text{const}$  are regular hypersurfaces, then, if the numerator of equation (23) does not vanish, we can conclude that  $K_m^m + \frac{\phi_{,\rho'}}{\phi} \neq 0$  for  $0 < \rho' < \varepsilon$ . By virtue of the continuous junction between the external and internal solutions at  $\rho' = \varepsilon$ , we have

$$\left[ K_m^m + \left( \frac{\phi, \rho'}{\phi} \right) \right]_{int} = \left[ K_m^m + \left( \frac{\phi, \rho'}{\phi} \right) \right]_{ext} = \frac{1-b}{\rho(\varepsilon)}. \quad (24)$$

Thus, we are left with three cases to be investigated:  $0 < b < 1$ ,  $b = 1$  and  $b > 1$ . As we shall see later, the case  $b > 1$  represents a distinctive feature of Brans-Dicke theory.

a) Case  $0 < b < 1$ . In this case we have  $K_m^m + \left( \frac{\phi, \rho'}{\phi} \right) > 0$  at the boundary of the tube.

Since this term does not vanish in the interior region, it does not change its sign. Hence, we conclude that  $K_m^m + \left( \frac{\phi, \rho'}{\phi} \right) > 0$  in the interval  $0 < \rho' < \varepsilon$ . On the other hand, it is easily shown that  $K_m^m + \left( \frac{\phi, \rho'}{\phi} \right) = \frac{\partial}{\partial \rho'} \ln \left( |\phi| \sqrt{-{}^{(2)}g} \right)$ . Therefore, for  $b < 1$  the function  $|\phi| \sqrt{-{}^{(2)}g}$  increases monotonically in the interval  $0 < \rho' < \varepsilon$ . Thus, we have

$$|\phi \mathcal{K}_\varphi^\varphi|_0 = \left( |\phi| \sqrt{-{}^{(2)}g} \right)_0 < \left( |\phi| \sqrt{-{}^{(2)}g} \right)_\varepsilon = |\psi| \sqrt{-{}^{(2)}h} [\rho(\varepsilon)]^{1-b}, \quad (25)$$

where  $\sqrt{-{}^{(2)}h} = \sqrt{-h_{tt}h_{zz}}$ . Since  $\rho(\varepsilon) \rightarrow 0$  when  $\varepsilon \rightarrow 0$ , we conclude that

$$\lim_{\varepsilon \rightarrow 0} |\phi \mathcal{K}_\varphi^\varphi|_0 = 0. \quad (26)$$

b) Case  $b > 1$ . Here the situation is exactly opposite to the former case. Since  $K_m^m + \left( \frac{\phi, \rho'}{\phi} \right) < 0$  the function  $|\phi| \sqrt{{}^{(2)}g}$  decreases monotonically for  $0 < \rho' < \varepsilon$ . Then,

$$|\phi \mathcal{K}_\varphi^\varphi|_0 = \left( |\phi| \sqrt{{}^{(2)}g} \right)_0 > \left( |\phi| \sqrt{{}^{(2)}g} \right)_\varepsilon = |\psi| \sqrt{{}^{(2)}h} [\rho(\varepsilon)]^{1-b}. \quad (27)$$

Therefore,

$$\lim_{\varepsilon \rightarrow 0} |\phi \mathcal{K}_\varphi^\varphi|_0 \rightarrow \infty. \quad (28)$$

Clearly, this case is unphysical and should be discarded, so  $b$  must be restricted to the interval  $0 < b \leq 1$ .

c) Case  $b = 1$ . We cannot apply the previous analysis when  $b = 1$ . Thus, let us use the following argument. Consider the interior metric and the scalar field written respectively in the form

$$ds^2 = d\rho'^2 + [M_\varepsilon(t, z) + m_\varepsilon(\rho', t, z)] dt^2 + [N_\varepsilon(t, z) + n_\varepsilon(\rho', t, z)] dz^2 + \\ [L_\varepsilon(t, z) + l_\varepsilon(\rho', t, z)] dt dz + P_\varepsilon(\rho', t, z) d\varphi^2, \quad (29a)$$

$$\phi(\rho', t, z) = \eta_\varepsilon(t, z) + \sigma_\varepsilon(\rho', t, z), \quad (29b)$$

where, by regularity conditions, we must have  $m_\varepsilon(0, t, z) = n_\varepsilon(0, t, z) = l_\varepsilon(0, t, z) = \sigma_\varepsilon(0, t, z) = 0$  and  $P_\varepsilon(\rho', t, z) \simeq \rho'^2$  in the limit  $\rho' \rightarrow 0$ . By virtue of continuity at  $\rho' = \varepsilon$  we have the relations

$$M_\varepsilon(t, z) + m_\varepsilon(\varepsilon, t, z) = h_{tt}(t, z)\rho(\varepsilon)^{2a} \quad (30a)$$

$$N_\varepsilon(t, z) + n_\varepsilon(\varepsilon, t, z) = h_{zz}(t, z)\rho(\varepsilon)^{2c} \quad (30b)$$

$$L_\varepsilon(t, z) + l_\varepsilon(\varepsilon, t, z) = 0 \quad (30c)$$

$$\eta_\varepsilon(t, z) + \sigma_\varepsilon(\varepsilon, t, z) = \psi(t, z)\rho(\varepsilon)^d \quad (30d)$$

Then, we have

$$\begin{aligned} \phi\mathcal{K}_\varphi^\varphi|_{\rho'=0} &= (\psi\rho(\varepsilon)^d - \sigma_\varepsilon(\varepsilon, t, z)) [(h_{tt}\rho(\varepsilon)^{2a} - m_\varepsilon(\varepsilon, t, z))(h_{zz}\rho(\varepsilon)^{2c} - n_\varepsilon(\varepsilon, t, z)) \\ &\quad - \frac{1}{4}(l_\varepsilon(\varepsilon, t, z))^2]^{\frac{1}{2}}. \end{aligned} \quad (31)$$

If we do not want the result above to depend explicitly on the details of the internal solution, then we must restrict ourselves to models of matter which  $m_\varepsilon, n_\varepsilon, l_\varepsilon$  and  $\sigma_\varepsilon$  go to zero as  $\varepsilon \rightarrow 0$  and satisfy the following convergence requirements:

$$\lim_{\varepsilon \rightarrow 0} \rho^{-2a}m_\varepsilon(\varepsilon, t, z) = \lim_{\varepsilon \rightarrow 0} \rho^{-2c}n_\varepsilon(\varepsilon, t, z) = \lim_{\varepsilon \rightarrow 0} \rho^d l_\varepsilon(\varepsilon, t, z) = \lim_{\varepsilon \rightarrow 0} \rho^{-d} \sigma_\varepsilon(\varepsilon, t, z) = 0. \quad (32)$$

Therefore, in this case

$$\lim_{\varepsilon \rightarrow 0} \phi\mathcal{K}_\varphi^\varphi|_0 = \psi\sqrt{-{}^{(2)}h}. \quad (33)$$

Let us now look into the equation (20b). We expect, by regularity requirements, that the scalar field  $\phi$  is finite in the axis  $L$  of the matter tube. It follows that  $\lim_{\rho' \rightarrow 0} \phi_{int}$  is finite. Whence we have  $\lim_{\rho' \rightarrow 0} (\rho'\phi_{int, \rho'}) = 0$ . Then, since  $\sqrt{-g} \simeq \sqrt{{}^{(2)}g}\rho'$  near the tube axis, we conclude that

$$\sqrt{-g}\phi_{,\rho'}|_0 = 0, \quad (34)$$

a result that holds for any value of  $b$ .

Taking the above results into account we now are able to find out the field equations for a simple line. The cases  $0 < b < 1$  and  $b = 1$  must be considered separately.

a) Case  $0 < b < 1$ . The equations (20a) and (20b) yield

$$\mathfrak{L}_j^i = -\frac{1}{4} \left[ \mathcal{C}_j^i - \delta_j^i \left( \frac{1+\omega}{\omega} \right) \mathcal{C} \right], \quad (35a)$$

$$\ell = \frac{4}{2\omega+3} \mathfrak{L}. \quad (35b)$$

The latter equation may be read as  $\ell = \frac{1}{\omega} \mathcal{C}$  and represents a further constraint on the functions  $a, b, c$  and  $d$ . Hence, we have the following set of constraints:

$$d = \frac{1}{1+\omega} \quad (36a)$$

$$a + b + c = \frac{\omega}{1+\omega} \quad (36b)$$

$$a^2 + b^2 + c^2 = \frac{\omega}{1+\omega} \quad (36c)$$

It turns out that the above system of algebraic equations can have real solutions only if  $\omega < -\frac{3}{2}$  or  $\omega > 0$ .

b) Case  $b = 1$ . From (21) e (33), we can write the equations (20a) e (20b) in the form

$$\mathfrak{L}_j^i = -\frac{1}{4} \left( [\mathcal{C}_j^i] - \delta_j^i \left( \frac{1+\omega}{\omega} \right) [\mathcal{C}] \right) \quad (37a)$$

$$\ell = \frac{4}{2\omega+3} \mathfrak{L} = \frac{1}{\omega} [\mathcal{C}], \quad (37b)$$

where  $[\mathcal{C}_j^i] = \mathcal{C}_j^i - \psi \sqrt{-{}^{(2)}h} \delta_\varphi^i \delta_j^\varphi$ . From the asymptotic form of the metric given by (13) a straightforward calculation yields  $\mathcal{C} = \psi \sqrt{-{}^{(2)}h} \lambda (1-d)$ , where we are defining  $\lambda \equiv \sqrt{h_{33}}$ . Now, from the equation  $\ell = \frac{1}{\omega} (\mathcal{C} - \psi \sqrt{-{}^{(2)}h})$  and taking into account the constraints (16) and (17) we have for  $b = 1$ , the following system of equations:

$$a + c = -\frac{1}{1+\omega} q \quad (38a)$$

$$a^2 + c^2 = -\frac{1}{1+\omega} q^2 \quad (38b)$$

$$d = \frac{1}{1+\omega} q \quad (38c)$$

with  $q \equiv (1 - \frac{1}{\lambda})$ .

This completes our analysis. Equations (35a), (35b),(37a) and (37b) together with the constraints relations represent in the context of Brans-Dicke the field equations describing the simple line  $L$ . In the next section we apply this formulation to the case of a static space-time with cylindrical symmetry.

#### IV. SIMPLE LINE SOURCES IN THE STATIC SPACE-TIMES WITH CYLINDRICAL SYMMETRY

It is known that the general form of the metric of a static space-time with cylindrical symmetry may be written in the form [13]

$$ds^2 = d\rho^2 - e^{A(\rho)}dt^2 + e^{B(\rho)}d\varphi^2 + e^{C(\rho)}dz^2, \quad (39)$$

where  $-\infty < t, z < \infty$ ,  $\rho > 0$  e  $0 < \varphi < 2\pi$ . For the scalar field we must have  $\phi = \phi(\rho)$  by virtue of symmetry.

Let us consider the space-time generated by a singular source concentrated along the axis  $\rho = 0$ . Thus, for  $\rho \neq 0$  the metric and the scalar field must satisfy Brans-Dicke field equations (5) in vacuum. The general solution for the metric and the scalar field is given by

$$ds^2 = d\rho^2 - \rho^{2a}dt^2 + \lambda^2\rho^{2b}d\varphi^2 + \rho^{2c}dz^2 \quad (40a)$$

$$\phi = \phi_0\rho^d, \quad (40b)$$

where  $a, b, c, d$  and  $\phi_0$  are constants that must satisfy the constraint equations (16) and (17).

It is reasonable assuming that, in the limit  $\omega \gg 1$ ,  $\phi_0$  has the limit [15]:

$$\lim_{\omega \rightarrow \infty} \phi_0 = \frac{1}{G}, \quad (41)$$

with  $G$  denoting the gravitational constant.

We see that (40a) coincides with the asymptotic form of the metric generated by a simple line when  $\rho \rightarrow 0$ . Therefore, to determine the energy-momentum tensor associated with the

source which generates the space-time (40a), two cases must be investigated separately:  $0 < b < 1$  or  $b = 1$ .

a) Case  $0 < b < 1$ . In this case it is easy to see that (35a) and (35b) yield

$$\mathfrak{L}_j^i = \frac{1}{4} \phi_0 \lambda \text{diag}(1 - a, 1 - b, 1 - c) \quad (42\text{a})$$

$$\phi = \phi_0 \rho^{\frac{1}{1+\omega}}, \quad (42\text{b})$$

with the constants  $a$ ,  $b$  and  $c$  satisfying the constraint relations (36).

At this point two remarks should be made. One refers to the fact that if  $|\omega| \rightarrow \infty$ , then the solution (40a) reduces to the General Relativity solution corresponding to Kasner vacuum metric which represents the gravitational field of an infinite rod [14]. The second remark concerns the possibility of the linear mass density of the source  $\mu \equiv -\mathfrak{L}_t^t$  being positive as it can happen for  $\omega < -2$ .

b) Case  $b = 1$ . From the equations (37) and setting  $b = 1$  in (40a), it follows:

$$[\mathcal{C}_j^i] = \phi_0 \lambda \text{diag}(a, q, c), \quad (43)$$

Hence,

$$\mathfrak{L}_j^i = -\frac{1}{4} \phi_0 \lambda \text{diag}(a - q, 0, c - q). \quad (44)$$

As we have mentioned earlier, in the case  $b = 1$ , the constants  $a$  e  $c$  must satisfy (38). Then, we can express  $a$  and  $c$  in terms of  $q$ :

$$a = -\frac{1/2}{1+\omega} \left( 1 \pm \sqrt{-(2\omega+3)} \right) q \quad (45\text{a})$$

$$c = -\frac{1/2}{1+\omega} \left( 1 \mp \sqrt{-(2\omega+3)} \right) q \quad (45\text{b})$$

$$d = \frac{1}{1+\omega} q \quad (45\text{c})$$

where we must impose  $\omega < -\frac{3}{2}$ , otherwise these constants will not be real numbers. Substituting (45) into (44) and recalling that  $\lambda q = \lambda - 1$  we obtain

$$\mathfrak{L}_j^i = -\frac{1}{4} (1 - \lambda) \text{diag}(\xi_{\pm}(\omega), 0, \xi_{\mp}(\omega)), \quad (46)$$

where we define

$$\xi_{\pm}(\omega) \equiv \frac{(2\omega + 3) \pm \sqrt{-(2\omega + 3)}}{2(1 + \omega)} \phi_0. \quad (47)$$

Thus, from (46) we conclude that the energy-momentum tensor must have the form

$$\mathcal{L}_j^i = \text{diag}(-\mu, 0, \tau_z), \quad (48)$$

with an equation of state given by

$$\frac{\tau_z}{\mu} = -\frac{\xi_{\mp}(\omega)}{\xi_{\pm}(\omega)}. \quad (49)$$

We can see that the metric induced on the surfaces  $t = \text{const}$  and  $z = \text{const}$  coincides with the metric of the cone, with the angular deficit given by the constant  $\lambda$ . From (46), a connection between the energy density of the line  $\mu$  and the angular deficit  $\lambda$  is established:

$$\lambda = 1 - \frac{4\mu}{\xi_{\pm}(\omega)}. \quad (50)$$

It is not difficult to see that in the case of the solution  $\xi_{-}(\omega)$  if we take  $-2 < \omega < -\frac{3}{2}$ , then rather than angular deficit we have an angular excess, even for a source with positive energy density. This situation is completely new and peculiar to Brans-Dicke theory in comparison to the results obtained in General Relativity [14].

Finally, it should be mentioned that in the light of the above results the well-known Gundlach and Ortiz cosmic string [4] is not included in the set of solutions considered previously. We shall discuss this point in the next section.

## V. THE GUNDLACH AND ORTIZ COSMIC STRING

As we have seen previously the Brans-Dicke field equations for line sources are given by (20) which by virtue of (21), (22) and (34) may be rewritten in the form

$$C_j^i - \delta_{\varphi}^i \delta_j^{\varphi} \zeta = -4 \left( \mathcal{L}_j^i - \frac{1}{2} f(\omega) \mathcal{L} \right) \quad (51a)$$

$$\ell = \frac{4}{3 + 2\omega} \mathcal{L} \quad (51b)$$

with  $\mathcal{C}_j^i$  and  $\ell$  given by (8) and (9) respectively, and

$$\zeta \equiv \lim_{\varepsilon \rightarrow 0} \eta \sqrt{-{}^{(2)}g_{int}} \Big|_{\rho'=0} \quad (52)$$

where  $g_{int}$  denotes the determinant of  $g_{mn}$  in the interior region and  $\eta = \lim_{\rho' \rightarrow 0} (\phi_{int})$  as defined in section III.

It turns out that from (51) one can relate the geometry and the scalar field generated in a region close to the line (which are present in the equations as  $\mathcal{C}_j^i$  and  $\ell$ ) with the physical properties of the source described by  $\mathcal{L}_j^i$ . However by only knowing  $\mathcal{L}_j^i$  one cannot from these equations determine the metric and scalar field due to the presence of  $\zeta$  which, as can be seen from its definition (52), depends on the internal structure of the source. Therefore, to work out the field equations (51) some informations concerning the internal matter distribution are needed, so that  $\zeta$  may be estimated. In this way simple lines would be classified according to the value of  $\zeta$  and, then, models corresponding to distinct classes would satisfy distinct field equations. In the previous section we have considered two distinct classes:

- 1) Case  $\zeta = 0$  (see eq. (26)). This corresponds to models for which  $b < 1$  and the numerator of equation (23) does not vanish in the interior region. As we have seen, these assumptions demand that  $\zeta = 0$ .
- 2) Case  $\zeta = \psi \sqrt{-{}^{(2)}h}$  (see eq. (33)). Here the models are required to satisfy the convergence conditions given by (32).

As far as the Gundlach and Ortiz cosmic string is concerned one can easily verify that it does not belong to any of the two cases mentioned above. In fact, one can show that the Gundlach and Ortiz solution corresponds to choosing  $\zeta = \psi \sqrt{-{}^{(2)}h}$  with  $b < 1$ . Clearly, this choice implies that the condition on the non-vanishing of the numerator of (23) must be relaxed.

As already seen, the metric of a static space-time with cylindrical symmetry and the scalar field which satisfy Brans-Dicke field equation in vacuum are given by (40), where the constants  $a, b, c$  e  $d$  must satisfy the constraint relations (16) and (17). Now, let us consider a line source described by  $\mathcal{L}_j^i = (-\mu, p_\varphi, -\mu)$ . We can verify that by choosing  $\zeta = \psi \sqrt{-{}^{(2)}h}$ ,

which reduces to  $\zeta = \phi_0$  for solution (40), since, in this case,  $\psi = \phi_0$  and  $\sqrt{-{}^{(2)}h} = 1$ , the field equations (51) yield

$$a = c \simeq \frac{4\mu}{3 + 2\omega} \frac{1}{\phi_0} \quad (53a)$$

$$b \simeq 1 - \left(\frac{4\mu}{\phi_0}\right)^2 \frac{1}{2 + 2\omega} \quad (53b)$$

$$d \simeq \frac{-8\mu}{3 + 2\omega} \frac{1}{\phi_0} \quad (53c)$$

$$\lambda \simeq 1 - \frac{4\mu}{\phi_0} \left(\frac{2 + 2\omega}{3 + 2\omega}\right) \quad (53d)$$

where by the symbol  $\simeq$  we are denoting the first correction with respect to linear energy density  $\mu$ . Further if the constraint (17) is to be satisfied then one is led to the following additional equation of state:

$$p_\varphi \simeq \frac{2\mu^2}{\phi_0} \frac{1}{1 + \omega} \quad (54)$$

In the limit  $\omega \gg 1$ , taken in Gundlach and Ortiz's paper, we expect that  $\phi_0 \simeq \frac{1}{G}$ , and so we obtain

$$a = c \simeq \frac{4\mu G}{3 + 2\omega} \quad (55a)$$

$$b \simeq 1 - \frac{(4\mu G)^2}{3 + 2\omega} \quad (55b)$$

$$d \simeq \frac{-8\mu G}{3 + 2\omega} \quad (55c)$$

$$\lambda \simeq 1 - 4\mu G \quad (55d)$$

At this stage, let us recall some points about the Gundlach and Ortiz's solution. Firstly, let us note that model assumed for the cosmic string is described by the action

$$\begin{aligned} \mathcal{S}_{matter} = & - \int d^4x \sqrt{-g} \left[ \frac{1}{2} [(\nabla_\nu \psi + ieA_\nu)\Phi] [(\nabla^\nu \psi + ieA^\nu)\Phi]^* \right. \\ & \left. \alpha (\Phi\Phi^* - \eta^2)^2 + \frac{1}{16\pi} F_{\nu\kappa} F^{\nu\kappa} \right], \end{aligned} \quad (56)$$

where  $A_\nu$  is a vector field,  $\Phi$  is a complex scalar field,  $\nabla_\nu$  is the covariant derivative with respect to the space-time metric,  $F_{\nu\kappa} \equiv \nabla_\nu A_\kappa - \nabla_\kappa A_\nu$ , and  $\alpha$ ,  $\eta$  and  $e$  are constants. Further it is assumed the ansatz

$$\Phi = \eta X(\rho) \exp(i\varphi) \quad (57)$$

$$A_\nu = \frac{1}{e} (P(\rho) - 1) \nabla_\nu \varphi \quad (58)$$

Secondly, according to a scheme of approximation the interior solution is characterized by the fact that effects of the scalar field are small in comparison with the cosmic string energy-momentum tensor contribution. Thus, in the interior region, the field equations would be approximately replaced by the Einstein equations. Therefore, one can make use of a result (see [17,18]) which states that, in the context of General Relativity, if one takes  $8\alpha = e^2$ , for the lower energy solution, the following relation is valid

$$\mu = \pi \eta^2 \quad (59)$$

Now, considering this result, we obtain after a coordinate transformation, the Gundlach and Ortiz's solution in the order of  $\eta^2$  :

$$ds^2 = r^{8\pi G \eta^2 \beta^2} \left[ -dt^2 + dz^2 + dr^2 + (1 - 4\pi G \eta^2)^2 d\varphi^2 \right] \quad (60)$$

$$\phi \sim r^{-8\pi G \eta^2 \beta^2} \quad (61)$$

where  $\beta^2 = \frac{1}{3+2\omega}$

## VI. THIN SHELLS IN BRANS-DICKE THEORY

The case of thin shells or surface layers was also considered by Israel [16], who presented a complete formulation of the problem. In this section we shall briefly outline an approach to the same problem in Brans-Dicke theory of gravity.

Let us consider the timelike hypersurface  $\Sigma$  which describes the history of a thin shell of matter in space-time. And let  $V$  be a neighborhood of  $\Sigma$  which admits a Gaussian coordinate system. In terms of these coordinates we can write the metric of space-time as

$$ds^2 = dn^2 + g_{ij}(n, x^k) dx^i dx^j, \quad (62)$$

where  $n$  is the coordinate associated with the vector  $\frac{\partial}{\partial n}$  normal to  $\Sigma$ , and  $i, j, k = 1, 2, 3$ . We can also choose the coordinate  $n$  such that  $n = 0$  corresponds to  $\Sigma \cap V$ .

Similarly to the decomposition (5) we can project Brans-Dicke field equations (4) onto and perpendicular to the hypersurfaces  $n = const$ . Then, in terms of the extrinsic and intrinsic curvatures of these hypersurfaces the field equations become identical to the equations (5) by just substituting the coordinate  $\rho$  for  $n$ , with  $K_j^i$  now denoting the extrinsic curvature of the hypersurface  $n = const$ .

Let us assume that the hypersurface  $\Sigma$  is smooth, with its geometry described by the tridimensional metric  $g_{ij}(n = 0, x^k)$ . We also suppose that the scalar field  $\phi$  is continuous on  $\Sigma$ , though its derivative  $\phi_{,n}$  may be discontinuous when  $\Sigma$  is crossed.

Now, if we integrate the field equations (5) with respect to the coordinate  $n$  in the interval  $-\varepsilon < n < \varepsilon$ , and take the limit  $\varepsilon \rightarrow 0$ , we easily obtain:

$$[K_j^i] = -\frac{8\pi}{\phi} \left( S_j^i - \frac{1}{2} \delta_j^i f(\omega) S \right) \quad (63a)$$

$$0 = S_\mu^n \quad (63b)$$

$$[\phi_{,n}] = \frac{8\pi}{3 + 2\omega} S, \quad (63c)$$

where

$$S_\nu^\mu \equiv \lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon}^{\varepsilon} T_\nu^\mu dn, \quad (64)$$

defines the energy-momentum tensor of the matter distribution concentrated on the shell and, as before, the bracket  $[X]$  of a quantity  $X$  denotes the operation

$$[X] \equiv \lim_{\varepsilon \rightarrow 0} [X|_{n=\varepsilon} - X|_{n=-\varepsilon}]. \quad (65)$$

Thus, we see that, as in the case of General Relativity, the equations (63) relate the physical properties of the matter lying on  $\Sigma$  to the geometry of space-time near  $\Sigma$ . From these equations we also see that when the matter distribution is not regular, and has a singular part with support on  $\Sigma$ , i.e.,  $T_\nu^\mu = (T_{reg})_\nu^\mu + S_\nu^\mu \delta(n)$ , then the singular part of the energy-momentum tensor causes a discontinuity to appear in the extrinsic curvature of  $\Sigma$ . In the case of Brans-Dicke theory this discontinuity depends also on the value the scalar field takes on  $\Sigma$  as well as on the parameter  $\omega$ . On the other hand, we see that  $S_j^i$  also induces a discontinuity on the derivative of the scalar field  $\phi_{,n}$ , whose value depends on the trace  $S$ .

Finally, let us note that when  $\omega \rightarrow \infty$  the equations (63) become identical to those of General Relativity provided that  $\phi \rightarrow G$  in this limit [16].

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